Derivatives: Chain Rule and Power Rule

Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f[g(x)]$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \qquad \text{or equivalently,} \qquad \frac{dy}{dx} = y' = f'[g(x)]g'(x).$$

In applying the Chain Rule, think of the opposite function f °g as having an inside and an outside part:

$$y = f[\underbrace{g(x)}_{\substack{u=g(x)\\ \text{inside}}}] = \underbrace{f(u)}_{\text{outside}}$$

General Power Rule a special case of the Chain Rule.

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx} \; or\ equivalently, \frac{d}{dx}[u^n] = nu^{n-1}u'$$

Examples: Find the derivative of each function given below.

1. $y = (x^2 - 3x + 5)^{25}$

Let $u = x^2 - 3x + 5$. Then $\frac{du}{dx} = 2x - 3$, $y = u^{25}$, $\frac{dy}{du} = 25u^{24}, and$
$\frac{dy}{dx} = (25u^{24})(2x - 3) = 25(x^2 - 3x + 5)^{24}(2x - 3)$

2. $\sqrt{3x^2 - 2x + 3}$

Let $u = 3x^2 - 2x + 3, Then \frac{du}{dx} = 6x - 2, y = \sqrt{u} = u^{\frac{1}{2}}, \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$
$\frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}}(6x - 2) = \frac{1}{2}(3x^2 - 2x + 3)^{-\frac{1}{2}} \times (6x - 2) = \frac{6x-2}{2\sqrt{3x^2-2x+3}}$

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3. $y = 5\sqrt[3]{x^2 + \sqrt{x^3}}$

Rewrite the expression: $y = 5\left(x^2 + x^{3/2}\right)^{1/3}$

Let $u = x^2 + x^{3/2}$. Then, $\frac{du}{dx} = 2x + \frac{3}{2}x^{1/2}$, $y = 5u^{1/3}, \frac{du}{dx} = \frac{5}{3}u^{-2/3}$, *and*

$$\frac{dy}{dx} = \left(\frac{5}{3}u^{-2/3}\right)\left(2x + \frac{3}{2}x^{1/2}\right) = \frac{5}{3}\left(x^2 + x^{3/2}\right)^{-2/3}\left(2x + \frac{3}{2}x^{\frac{1}{2}}\right)$$

4. $y = e^{3x^2-5x+5}$

Let $u = 3x^2 - 5x + 5$. Then, $\frac{du}{dx} = 6x - 5$, $y = e^u$, $\frac{dy}{du} = e^u$, and

$\frac{dy}{dx} = e^u \times (6x + 5) = e^{3x^2-5x+5} \times (6x + 5)$

5. $\ln(2x^2 + 3x)$

Let $u = 2x^2 + 3x$, Then $\frac{du}{dx} = 4x + 3$, $y = \ln u, \frac{dy}{du} = \frac{1}{u}$, and

$\frac{dy}{dx} = \frac{1}{u} \times (4x + 3) = \frac{4x+3}{2x^2+3x}$

6. $y = \ln(\cos x^2)$.

Let $u = \cos x^2$ Then $\frac{du}{dx} = (-\sin x^2)(2x)$, $y = \ln u$, $\frac{dy}{du} = \frac{1}{u}$, and

$\frac{dy}{dx} = \left(\frac{1}{u}\right)(-\sin x^2)(2x) = -\frac{2x\sin x^2}{\cos x^2} = -2x\tan x^2$

This example illustrates how the Chain Rule is to be used **<u>all the way through.</u>**